AN EXPLICIT CONSTRUCTION OF CASIMIR OPERATORS AND EIGENVALUES : II

H. R. Karadayi and M. Gungormez
Dept.Physics, Fac. Science, Tech.Univ.Istanbul
80626, Maslak, Istanbul, Turkey
Internet: karadayi@sariyer.cc.itu.edu.tr

Abstract

It is given a way of computing Casimir eigenvalues for Weyl orbits as well as for irreducible representations of Lie algebras. A $\kappa(s)$ number of polinomials of rank N are obtained explicitly for A_N Casimir operators of order s where $\kappa(s)$ is the number of partitions of s into positive integers except 1. It is also emphasized that these eigenvalue polinomials prove useful in obtaining formulas to calculate weight multiplicities and in explicit calculations of the whole cohomology ring of Classical and also Exceptional Lie algebras.

I. INTRODUCTION

In a previous paper [**1**] which we refer (I) throughout the work, we establish the **most general** explicit forms of 4th and 5th order Casimir operators of A_N Lie algebras. By starting from this point, we want to develop a framework which makes possible to calculate, for the irreducible representations of A_N Lie algebras, the eigenvalues of Casimir operators in any order. Extensions are also possible to any other classical or exceptional Lie algebra because any Lie algebra has always an appropriate subalgebra of type A_N.

For a Casimir operator I(s) of degree s, the eigenvalues for a D-dimensional representation are known to be calculated in the following form:

$$\frac{1}{D} \quad Trace(I(s)) \quad . \qquad (I.1)$$

A direct calculation of (I.1) could become problematic in practice as the dimension of representation grows high. Additionally to the ones given in (I), we give here some further works [**2**] dealing with this problem.

A second essential problem arisen here is due to the fact that one must also calculate weight multiplicities for representations comprising more than one Weyl orbit. This latter problem is known to be solved by formulas which are due to Kostant and Freudenthal [**3**] and it is at the root of Weyl-Kac character formulas [**4**]. Although they are formally explicit, these two formulas are of recursive character and hence they exhibit problems in practical calculations. One could therefore prefers to obtain a **functional formula** in calculating weight multiplicities. This will be dealt in a subsequent paper.

It is known, on the other hand, that **trace operations** can be defined [**5**] in two equivalent ways one of which is nothing but the explicit matrix trace. An expression like (I.1) could therefore not means for a Weyl orbit, **in general**. We instead want to extend the concept of Casimir eigenvalue to Weyl orbits. As we have introduced in an earlier work [**6**], we replace (I.1) with the following **formal** definition:

$$ch_s(\Pi) \equiv \sum_{\mu \in \Pi} (\mu)^s \qquad (I.2)$$

where Π is a Weyl orbit and the sum is over all weights μ included within Π. The powers of weights in (I.2) are to be thought of as s-times products

$$(\mu)^s = \overbrace{\mu \times \mu \times ... \times \mu}^{s\ times} \quad .$$

Note here that (I.2) is defined not only for Weyl orbits or representations but it means also for any collection of weights. We will mainly show in what follows how (I.2) gives us a way to obtain eigenvalues of a Casimir operator. Due to a permutational lemma given in section (II), the procedure works out especially for A_N Lie algebras. It will however be seen in a subsequent paper that it is generalized to any Classical or Exceptional Lie algebra. In section (III), we will give a general formula of calculating $ch_s(\Pi)$ by the aid of this permutational lemma. An efficient way of using this formula is due to reduction rules which are explained in section (IV) and the polinomials representing Casimir eigenvalues will be given in section (V) and also in appendix.2. We will show in section (VI) that the two formula (I.1) and (I.2) are in fact in coincidence.

II. A PERMUTATIONAL LEMMA FOR A_N WEYL ORBITS

In this section, we give, for A_N Lie algebras, a permutational lemma which says that, **modulo permutations, there is one-to-one correspondence between the Weyl chamber and the Tits cone [7]**. As will be explained below, such a correspondence appears only when one reformulates everything in terms of the so-called **fundamental weights**.

For an excellent study of Lie algebra technology we refer the book of Humphreys [**8**]. We give, however, some frequently used concepts here. In describing the whole **weight lattice** of a Lie algebra of rank N, the known picture will be provided by **simple roots** α_i and **fundamental dominant weights** λ_i where

indices like $i_1, i_2, ..$ take values from the set $I_\circ \equiv \{1, 2, ..N\}$. Any dominant weight Λ^+ can then be expressed by

$$\Lambda^+ = \sum_{i=1}^{N} r_i \ \lambda_i \quad , \quad r_i \in Z^+ \tag{II.1}$$

where Z^+ is the set of positive integers including zero. We know that a Weyl orbit Π is stable under the actions of Weyl group of Lie algebra. This means that all weights within a Weyl orbit are equivalent under the actions of Weyl group and they can be obtained from any one of them by performing Weyl conjugations one-by-one. We thus obtain a description of the whole weight lattice of which any weight is given by

$$\mu = m_1 \ \lambda_1 + m_2 \ \lambda_2 + ... + m_N \ \lambda_N \quad , \quad \pm m_i \in Z^+ \quad . \tag{II.2}$$

Our way of thinking of a Weyl orbit is, on the other hand, based on the fact that **Weyl reflections can be replaced by permutations** for A_N Lie algebras. It is seen in the following that essential figures for this are **fundamental weights** $\mu_{\mathbf{I}}$ which we introduced **[9]** some fifteen years ago:

$$\begin{aligned} &\mu_1 \equiv \lambda_1 \\ &\mu_i \equiv \mu_{i-1} - \alpha_{i-1} \quad , \quad i = 2, 3, ..N + 1. \end{aligned} \tag{II.3}$$

Indices like $I_1, I_2, ...$ take values from the set $S_\circ \equiv \{1, 2, ..N, N + 1\}$. Recall here that the weights defined in (II.3) are nothing but the weights of (N+1)-dimensional fundamental representations of A_N Lie algebras. To prevent confusion, note here that some authors prefer to call λ_i's fundamental weights.Though there are N+1 number of fundamental weights μ_I, they are not completely linear independent due to the fact that their sum is zero. The main observation is, however, that (II.2) replaces with

$$\mu = q_1 \ \mu_{I_1} + q_2 \ \mu_{I_2} + .. + q_{N+1} \ \mu_{I_{N+1}} \tag{II.4}$$

when one reformulates in terms of N+1 fundamental weights. The conditions

$$I_1 \neq I_2 \neq .. \neq I_{N+1} \tag{II.5}$$

must be taken into account for each particular weight (II.4) and one can always assume that

$$q_1 \geq q_2 \geq ... \geq q_{N+1} \geq 0 \quad . \tag{II.6}$$

(II.6) receives here further importance in the light of following lemma:

Let *P(N)* be the weight lattice of A_N Lie algebra. A dominant weight $\Lambda^+ \in P(N)$ has always the form of

$$\Lambda^+ = q_1 \ \mu_1 + q_2 \ \mu_2 + .. + q_{N+1} \ \mu_{N+1} \tag{II.7}$$

and hence the whole Weyl orbit $\Pi(\Lambda^+)$ is obtained by permutations of (II.7) over N+1 fundamental weights. In the basis of fundamental weights all weights of the Weyl orbit $\Pi(\Lambda^+)$ are thus seen in the common form (II.4) where all indices I_k take values from the set $S_\circ$ together with the conditions (II.5).

Although it is not in the scope of this work, demonstration of lemma is a direct result of the definitions (II.3). It will be useful to realize the lemma further in terms of (N+1)-tuples which re-define (II.7) in the form

$$\Lambda^+ \equiv (q_1, q_2, ..q_{N+1}) \quad . \tag{II.8}$$

Then every elements $\mu \in \Pi(\Lambda^+)$ corresponds to a permutation of $q_i's$:

$$\mu = (q_{I_1}, q_{I_2}, .., q_{I_{N+1}})$$

.

To this end, let us choose a weight

$$-\lambda_1 + 2 \ \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 + \lambda_7 \tag{II.9}$$

which is expressed in the conventional form (II.2). By taking inverses

$$\lambda_i \equiv \mu_1 + \mu_2 + ... + \mu_i \quad , \quad i \in I_\circ \qquad (II.10)$$

of (II.2), we can re-express (II.9) as

$$2\mu_1 + 3\ \mu_2 + \mu_3 + 2\ \mu_4 + \mu_5 + \mu_7 \qquad (II.11)$$

which says us that

$$-\lambda_1 + 2\ \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 + \lambda_7 \in \Pi(\lambda_1 + \lambda_3 + \lambda_6) \ \ .$$

It is obvious that this last knowledge is not so transparent in (II.9).

One must further emphasize that the lemma allows us to know the dimensions of Weyl orbits directly from their dominant representatives. For this and further use, let us re-consider (II.1) in the form

$$\Lambda^+ \equiv u_1\ \lambda_{i_1} + u_2\ \lambda_{i_2} + ... + u_\sigma\ \lambda_{i_\sigma} \quad , \quad u_\sigma \in Z^+ - 0 \qquad (II.12)$$

with

$$i_1 \leq i_2 \leq ... \leq i_\sigma \quad , \quad \sigma = 1, 2, ..N. \ \ . \qquad (II.13)$$

Then, it is seen that the number of weights within a Weyl orbit $\Pi(\Lambda^+)$ is

$$dim\Pi(\Lambda^+) = \frac{(N+1)!}{\xi(\Lambda^+)\ (N+1-i_\sigma)!} \qquad (II.14)$$

where

$$\xi(\Lambda^+) \equiv \prod_{j=1}^{\sigma} (i_j - i_{j-1})! \quad , \quad i_0 \equiv 0 \ \ . \qquad (II.15)$$

We therefore assume in the following that $dim\Pi(\Lambda)$ is always known to be a polinomial of rank N.

III. EIGENVALUES FOR WEYL ORBITS

As is mentioned above, eigenvalues are, in fact, known to be defined for representations. A representation $R(\Lambda^+)$ is, on the other hand, determined from its **orbital decomposition**:

$$R(\Lambda^+) = \Pi(\Lambda^+) \quad + \sum_{\lambda^+ \in Sub(\Lambda^+)} m(\lambda^+ < \Lambda^+)\ \ \Pi(\lambda^+) \qquad (III.1)$$

where $Sub(\Lambda^+)$ is the set of all sub-dominant weights of Λ^+ and $m(\lambda^+ < \Lambda^+)$'s are multiplicities of weights λ^+ within the representation $R(\Lambda^+)$. Once a convenient definition of eigenvalues is assigned to $\Pi(\lambda^+)$ for $\lambda^+ \in Sub(\Lambda^+)$, it is clear that this also means for the whole $R(\Lambda^+)$ via (III.1). In the rest of this section, we then show how definition (I.2) can be used to obtain **orbit eigenvalues** as N-dependent polinomials.

Let us now make some definitions which are used frequently for description of **symmetric polinomials** encountered in the root expansions which take place heavily in the recently studied electromagnetically dual supersymmetric theories **[10]**. These will, of course, be given here in terms of fundamental weights μ_I. The essential role will be played by **generators**

$$\mu(s) \equiv \sum_{I=1}^{N+1} (\mu_I)^s \quad , \quad s = 1, 2, ... \qquad (III.2)$$

and their reductive generalizations

$$\mu(s_1, s_2, .., s_k) \equiv \sum_{I_1, I_2, ..I_k = 1}^{N+1} (\mu_{I_1})^{s_1} (\mu_{I_2})^{s_2} ... \ (\mu_{I_k})^{s_k} \quad . \qquad (III.3)$$

For (III.3), the conditions

$$s_1 \geq s_2 \geq ... \geq s_k \qquad (III.4)$$

are always assumed and no two of indices $I_1, I_2, ..I_k$ shall take the same value for each particular monomial. Note also that $\mu(s, 0, 0, ..0) = \mu(s)$.

As the first step, we now make the suggestion, in view of (I.2), that orbit eigenvalues can be conveniently calculated by decomposing $ch_s(\Pi)$ in terms of quantities defined in (III.3) and this provides us the possibility to calculate orbit eigenvalues with the same ability regardless

(i) the rank N of algebra,

(ii) the dimension $dimR(\Lambda^+, N)$ of irreducible representation,

(iii) the order s of Casimir element.

To give our results below, we will assume that the set

$$s/k \equiv \{s_1, s_2, ..., s_k\} \qquad (III.5)$$

represents, via (III.4), all partitions

$$s = s_1 + s_2 + ... + s_k \quad , \quad s \geq k$$

of positive integer s to k-number of positive integers $s_1, s_2, ..s_k$. It is useful to remark here that each particular partition participating within a s/k gives us, modulo (N+1), a dominant weight in P(N) and the whole subdominant chain $Sub(s\ \lambda_1)$ is in one-to-one correspondence with the partitions within a s/k. This must always be kept in mind in the following considerations.

On the other hand, instead of (II.1), it is crucial here to use (II.7) in the form

$$\Lambda^+ \equiv \sum_{i=1}^{\sigma} q_i\ \mu_i \qquad (III.6)$$

where $\sigma = 1, 2, ..N + 1$. Note here that this is another form of (II.12). Due to permutational lemma given above, we now know that all weights of a Weyl orbit are specified with the same parameters $q_i, (i = 1, 2, ..\sigma)$. It is only of this fact which allows us to obtain the following formula in expressing orbital eigenvalues:

$$\Omega_s(q_1, q_2, ..., q_\sigma, N) = \frac{1}{(N + 1 - \sigma)!} \sum_{k=1}^{\sigma} (N + 1 - k)!\ \xi(s/k)\ Factors(s/k) \qquad (III.7)$$

where we define, for all possible partitions (s/k),

$$Factors(s/k) \equiv M(s_1, s_2, ..., s_k)\ q(s_1, s_2, ..., s_k)\ \mu(s_1, s_2, ..., s_k) \qquad (III.8)$$

and the multinomial

$$M(s_1, s_2, ..., s_k) \equiv \frac{(s_1 + s_2 + ... + s_k)!}{s_1! s_2! ... s_k!}$$

together with the condition that

$$M(s_1, s_2, ..., s_k) \equiv 0 \quad for \quad s < k. \qquad (III.9)$$

$\xi(s/k)$ here is defined as in (II.15) because, as we remark just above, any permutation within a s/k determines a dominant weight. As in exactly the same way in (III.3), we also define

$$q(s_1, s_2, .., s_k) \equiv \sum_{s_1, s_2, ..s_k = 1}^{\sigma} (q_{I_1})^{s_1} (q_{I_2})^{s_2} ... (q_{I_k})^{s_k} \quad . \qquad (III.10).$$

After all, one obtains a direct way to compute (I.2) in the form

$$ch_s(\Lambda^+, N) = \frac{1}{\xi(\Lambda^+)}\ \Omega_s(q_1, q_2, ..., q_k, N) \qquad (III.11)$$

for all $q_1 \geq q_2 \geq .. \geq q_k$. For cases which we consider in this work, we will give in appendix.1 some exemplary expressions extracted from (III.7).

IV. REDUCTION FORMULAS

Although it has an explicit form, the simplicity of formula (III.7) is not so transparent to an experienced eye looking for its advanced applications. This point can be recovered by recursively reducing the quantities (III.9) up to generators $\mu(s)$ defined in (III.2). We call these **reduction rules**. We will only give the ones which we need in the sequel. It would however be useful to mention about some of their general features. As is known, elementary Schur functions $\mathbf{S_k(x)}$ are defined by expansions

$$\sum_{k\in Z^+} S_k(x)\ z^k \equiv exp\sum_{k=1}^{\infty} x_k\ z^k \qquad (IV.1)$$

with the following explicit expressions:

$$S_k(x) = \sum_{k_1+2\ k_2+3\ k_3..=k} \frac{x_1^{k_1}}{k_1!}\ \frac{x_2^{k_2}}{k_2!}... \quad , \quad k>0\ . \qquad (IV.2)$$

The complete symmetric functions $h_k(\mu_1,\mu_2,..\mu_N)$ are defined, on the other hand, by

$$\prod_{i=1}^{N} \frac{1}{(1-z\ \mu_i)} \equiv \sum_{k\geq 0} h_k(\mu_1,\mu_2,..\mu_N)\ z^k \quad . \qquad (IV.3)$$

It can be easily shown that the known equivalence

$$h_k(\mu_1,\mu_2,..\mu_N) \equiv S_k(x) \qquad (IV.4)$$

is now conserved by the reduction rules with the aid of a simple replacement

$$\mu(s) \rightarrow s\ x_s \quad .$$

A simple but instructive example concerning (IV.4) for k=4 is

$$h_4(\mu_1,\mu_2,\mu_3,\mu_4) = \mu(4)+\mu(3,1)+\mu(2,2)+\mu(2,1,1)+\mu(1,1,1,1) \qquad (IV.5)$$

with the corresponding reduction rules

$$\begin{aligned} q(1,1,1,1) &= \frac{1}{24}\ q(1)^4 - \frac{1}{4}\ q(1)^2\ q(2) + \frac{1}{8}\ q(2)^2 + \frac{1}{3}\ q(1)\ q(3) - \frac{1}{4}\ q(4) \quad , \\ q(2,1,1) &= \frac{1}{2}\ q(1)^2 q(2) - \frac{1}{2}\ q(2)^2 - q(1)\ q(3) + q(4) \quad , \\ q(3,1) &= q(1)\ q(3) - q(4) \quad , \\ q(2,2) &= \frac{1}{2}\ q(2)^2 - \frac{1}{2}\ q(4) \quad . \end{aligned} \qquad (IV.6)$$

For other cases of interest, the reduction rules will be given in appendix.1 respectively for the partitions of 5,6 and 7.

V. EXISTENCE OF EIGENVALUE POLINOMIALS

After all these preparations, we are now in the position to bring out the most unexpected part of work. This is the possibility to extend (III.11) directly for irreducible representations as well as Weyl orbits. We will show in a subsequent work that this gives us the possibility to obtain infinitely many functional formulas to calculate weight multiplicities and also to make explicit calculations of nonlinear cohomology relations which are known to be exist **[11]** for classical and exceptional Lie algebras.

In view of the fact that $\mu(1) \equiv 0$, one can formally decompose (III.11) in the form

$$ch_s(\Lambda^+, N) \equiv \sum_{s/k} cof_{s_1 s_2 .. s_k}(\Lambda^+, N)\ \mu(s_1)\mu(s_2)..\mu(s_k) \qquad (V.1)$$

and this allows us to define a number of polinomials

$$P_{s_1 s_2 .. s_k}(\Lambda^+, N) \equiv \frac{cof_{s_1 s_2 .. s_k}(\Lambda^+, N)}{cof_{s_1 s_2 .. s_k}(\lambda_k, N)}\ \frac{dimR(\lambda_k, N)}{dimR(\Lambda^+, N)}\ P_{s_1 s_2 .. s_k}(\lambda_k, N) \quad . \qquad (V.2)$$

Note here that

$$cof_{s_1 s_2 .. s_k}(\lambda_i, N) \equiv 0 \quad , \quad i < k \qquad (V.3)$$

and also

$$dimR(\lambda_i, N) = M(N+1, i) \quad , \quad i = 1, 2, ..N. \quad . \qquad (V.4)$$

To proceed further, we will work on the explicit example of 4th order for which (V.1) and (V.2) give

$$ch_4(\Lambda^+, N) \equiv cof_4(\Lambda^+, N)\ \mu(4) + cof_{22}(\Lambda^+, N)\ \mu(2)^2 \quad , \qquad (V.5)$$

$$P_4(\Lambda^+, N) \equiv \frac{cof_4(\Lambda^+, N)}{cof_4(\lambda_1, N)}\ \frac{dimR(\lambda_1, N)}{dimR(\Lambda^+, N)}\ P_4(\lambda_1, N) \quad , \qquad (V.6)$$

and

$$P_{22}(\Lambda^+, N) \equiv \frac{cof_{22}(\Lambda^+, N)}{cof_{22}(\lambda_2, N)}\ \frac{dimR(\lambda_2, N)}{dimR(\Lambda^+, N)}\ P_{22}(\lambda_2, N) \qquad (V.7)$$

N dependences are explicitly written above. The main observation here is to change the variables r_i of (II.1):

$$1\ +\ r_i \equiv \theta_i\ -\ \theta_{i+1} \qquad (V.8)$$

and to suggest the decompositions

$$\begin{aligned} P_4(\Lambda^+, N) = & k_4(1, N)\ \Theta(4, \Lambda^+, N)\ + \\ & k_4(2, N)\ \Theta(2, \Lambda^+, N)^2 + \\ & k_4(3, N)\ \Theta(3, \Lambda^+, N)\ + \\ & k_4(4, N)\ \Theta(2, \Lambda^+, N)\ + \\ & k_4(5, N) \end{aligned} \qquad (V.9)$$

and

$$\begin{aligned} P_{22}(\Lambda^+, N) = & k_{22}(1, N)\ \Theta(4, \Lambda^+, N)\ + \\ & k_{22}(2, N)\ \Theta(2, \Lambda^+, N)^2 + \\ & k_{22}(3, N)\ \Theta(3, \Lambda^+, N)\ + \\ & k_{22}(4, N)\ \Theta(2, \Lambda^+, N)\ + \\ & k_{22}(5, N) \end{aligned} \quad . \qquad (V.10)$$

As in (III.2) or (III.10), we also define here the generators

$$\Theta(s, \Lambda^+, N) \equiv \sum_{i=1}^{N+1} (\theta_i)^s \quad . \qquad (V.11)$$

It is seen then that (V.9) and (V.10) are the most general forms compatible with $\Theta(1, \Lambda^+, N) \equiv 0$. What is significant here is the possibility to solve equations (V.6) and (V.7) in view of assumptions (V.9) and (V.10)

but with coefficients $k_4(\alpha, N), k_{22}(\alpha, N)$ **which are independent of** Λ^+ for $\alpha = 1, .., 5$. By examining for a few simple representations, one can easily obtain the following **non-zero** solutions for these coefficients:

$$\begin{aligned} k_4(1,N) &= \frac{720}{g_4(N)} \; (N^2+2N+2) \; k_4(5,N) \\ k_4(2,N) &= - \; \frac{720}{g_4(N) \; (N+1)} \; (2 \; N^2 + 4 \; N - 1) \; k_4(5,N) \end{aligned} \qquad (V.14)$$

and

$$\begin{aligned} k_{22}(1,N) &= - \; \frac{1440}{g_{22}(N)} \; (2 \; N^2 + 4 \; N - 1) \; k_{22}(5,N) \\ k_{22}(2,N) &= \frac{720}{g_{22}(N) \; (N+1)} \; (N^4 + 4 \; N^3 - 8 \; N + 13) \; k_{22}(5,N) \\ k_{22}(4,N) &= - \; \frac{120}{g_{22}(N)} \; (N-2) \; (N-1) \; (N+1)^2 \; (N+3) \; (N+4) \; k_{22}(5,N) \end{aligned} \qquad (V.15)$$

where

$$\begin{aligned} g_4(N) &\equiv \prod_{i=-2}^{4} (N+i) \\ g_{22}(N) &\equiv g_4(N) \; (5 \; N^2 + 10 \; N + 11) \quad . \end{aligned} \qquad (V.16)$$

The calculations goes just in the same way for orders 5,6 and 7 and hence we directly give our solutions in appendix.2.

VI. CONCLUSIONS

In (I), we have obtained the most general formal operators representing 4th and also 5th order Casimir invariants of A_N Lie algebras. By comparing with the ones appearing in litterature, they are the most general in the sense that both are to be expressed in terms of two free parameters. As is shown in (I), all coefficient polinomials of 4th order Casimir operators are expressed in terms of u(1) and u(2) while those of 5th order Casimirs are v(1) and v(2). As is also emphasized there, the existence of two free parameters for both cases can be thought of as related with the partitions 4=2+2 and 5=3+2. Recall here the polinomials P_4 and P_{22}. This gives us the possibility to calculate the trace forms (I.1) directly in any matrix representation of A_N Lie algebras. These trace calculations are straigtforward and show that eigenvalues of 4th order Casimir operators have the form of an explicit polinomial which depends on the rank N and two free parameters u(1) and u(2). It is thus seen that there are always appropriate choices of parameters u(1) and u(2) in such a way that this same polinomial reproduces $P_4(\Lambda^+, N)$ or $P_{22}(\Lambda^+, N)$ as given in (V.9) and (V.10). The same is also true for 5th order Casimirs. With the appropriate choice

$$k_4(5,N) \equiv \frac{1}{6!} \; (N+1)^2 \; (N+2) \; (N+3) \; (N+4) \qquad (VI.1)$$

in (V.9) it is sufficient to take

$$u(1) = 1 \quad , \quad u(2) = \frac{3 \; N - 8}{3 \; N} \qquad (VI.2)$$

in order to reproduce

$$\frac{1}{D} \; Trace(I(4)) \equiv P_4(\Lambda^+, N)$$

with $dimR(\Lambda^+, N) = D$. The data for other cases of interest are

$$\begin{aligned} k_{22}(5,N) &\equiv \frac{1}{6!} \; (5 \; N^2 + 10 \; N + 11) \; (N+1) \; (N+2) \; (N+3) \; (N+4) \\ u(1) = 1 \quad , \quad u(2) &= \frac{2}{3} \; \frac{2 \; N^2 + N + 2}{N \; (N+1)} \end{aligned} \qquad (VI.3)$$

for

$$\frac{1}{D}\ Trace(I(4)) \equiv P_{22}(\Lambda^+, N) \quad ,$$

and

$$\begin{aligned} k_5(2,N) &\equiv -5\ \frac{(N+1)\ (N^2+2\ N-1)}{N\ (N-1)\ (N-2)\ (N-3)} \\ v(1) = 1 \quad &, \quad v(2) = \frac{2\ N-5}{2\ N} \end{aligned} \tag{VI.4}$$

for

$$\frac{1}{D}\ Trace(I(5)) \equiv P_5(\Lambda^+, N) \quad ,$$

and

$$\begin{aligned} k_{32}(5,N) &\equiv -\frac{1}{12}\ \frac{(N+1)^3\ (N+4)\ (N+5)}{N\ (N-1)} \\ v(1) = 1 \quad &, \quad v(2) = \frac{(11\ N+5)\ (N-1)}{10\ N\ (N+1)} \end{aligned} \tag{VI.5}$$

for

$$\frac{1}{D}\ Trace(I(5)) \equiv P_{32}(\Lambda^+, N) \quad .$$

Now it is clear that, this would be a **direct evidence** for equivalence between the formal expressions (I.1) and (I.2). In result, it is seen that one can obtain $\kappa(s)$ number of different polinomials $P_{s_1,s_2,..s_k}(\Lambda^+, N)$ representing eigenvalues of A_N Casimir operators I(s) of order s, with $\kappa(s)$ is the number of partitions of s to all positive integers except 1. As is known from (I), this is just the number of free parameters to describe the most general form of I(s).

REFERENCES

[1] Karadayi H.R and Gungormez M: Explicit Construction of Casimir Operators and Eigenvalues:I , submitted to J.Math.Phys.

[2] Braden H.W ; Jour.Math.Phys. 29 (1988) 727-741 and 2494-2498
Green H.S and Jarvis P.D ; Jour.Math.Phys. 24 (1983) 1681-1687

[3] Kostant B. ; Trans.Am.Math.Soc. 93 (1959) 53-73
Freudenthal H. ; Indag.Math. 16 (1954) 369-376 and 487-491
Freudenthal H. ; Indag.Math. 18 (1956) 511-514

[4] Kac.V.G ; Infinite Dimensional Lie Algebras, 3rd edition, Cambridge University Press

[5] the paragraph (3.29) in Carter, R.W: Simple Groups of Lie Type, J.Wiley and sons (1972) N.Y

[6] Karadayi H.R, Jour.Math.Phys. 25 (1984) 141-144

[7] the section 3.12 in ref.4

[8] Humphreys J.E: Introduction to Lie Algebras and Representation Theory , Springer-Verlag (1972) N.Y.

[9] Karadayi H.R: Anatomy of Grand Unifying Groups , ICTP preprints (unpublished), IC/81/213 and 224

[10] Kutasov D, Schwimmer A and Seiberg N: Chiral rings, Singularity Theory and Electric-Magnetic Duality , **hep-th/9510222**

[11] Borel A and Chevalley C: Mem.Am.Math.Soc. 14 (1955) 1
Chih-Ta Yen: Sur Les Polynomes de Poincare des Groupes de Lie Exceptionnels, Comptes Rendue Acad.Sci. Paris (1949) 628-630
Chevalley C: The Betti Numbers of the Exceptional Simple Lie Groups, Proceedings of the International Congress of Mathematicians, 2 (1952) 21-24
Borel A: Ann.Math. 57 (1953) 115-207
Coleman A.J: Can.J.Math 10 (1958) 349-356

APPENDIX. 1

In this work, we consider the calculation of eigenvalues for A_N Casimir operators of orders s=4,5,6,7. It is however apparent that all our results are to be accomplished as in exactly the same way and with the same ability for all orders. The following applications of the formula (III.7) will be instructive for all other cases of interest:

$$\begin{aligned}\Omega_4(q_1,N) = \frac{1}{(N+1-1)!}\ &(\\ &1!\ (N+1-1)!\ M(4)\ q(4)\ \mu(4)\ \ )\ \ ,\end{aligned} \tag{A1.1}$$

$$\begin{aligned}\Omega_4(q_1,q_2,N) = \frac{1}{(N+1-2)!}\ &(\\ &1!\ (N+1-1)!\ M(4)\ q(4)\ \mu(4)\ + \\ &1!\ (N+1-2)!\ M(3,1)\ q(3,1)\ \mu(3,1)\ + \\ &2!\ (N+1-2)!\ M(2,2)\ q(2,2)\ \mu(2,2)\ \ )\ \ ,\end{aligned} \tag{A1.2}$$

$$\begin{aligned}\Omega_4(q_1,q_2,q_3,N) = \frac{1}{(N+1-3)!}\ &(\\ &1!\ (N+1-1)!\ M(4)\ q(4)\ \mu(4)\ + \\ &1!\ (N+1-2)!\ M(3,1)\ q(3,1)\ \mu(3,1)\ + \\ &2!\ (N+1-2)!\ M(2,2)\ q(2,2)\ \mu(2,2)\ + \\ &2!\ (N+1-3)!\ M(2,1,1)\ q(2,1,1)\ \mu(2,1,1)\ \ )\ \ ,\end{aligned} \tag{A1.3}$$

and for $k \geq 4$

$$\begin{aligned}\Omega_4(q_1,q_2,..,q_\sigma,N) = \frac{1}{(N+1-\sigma)!}\ &(\\ &1!\ (N+1-1)!\ M(4)\ q(4)\ \mu(4)\ + \\ &1!\ (N+1-2)!\ M(3,1)\ q(3,1)\ \mu(3,1)\ + \\ &2!\ (N+1-2)!\ M(2,2)\ q(2,2)\ \mu(2,2)\ + \\ &2!\ (N+1-3)!\ M(2,1,1)\ q(2,1,1)\ \mu(2,1,1)\ + \\ &4!\ (N+1-4)!\ M(1,1,1,1)\ q(1,1,1,1)\ \mu(1,1,1,1))\ \ .\end{aligned} \tag{A1.4}$$

On the other hand, for an effective application of (III.7), it is clear that one needs to reduce the generators $q(s_1,s_2,..s_k)$ in terms of $q(s)$'s. Following ones are sufficient within the scope of this work. Together with the condition that $\mu(1) \equiv 0$, the similar ones are valid also for $\mu(s_1,s_2,..s_k)$'s:

$$\begin{aligned}
&q(4,1) = q(1)\ q(4) - q(5)\\
&q(3,2) = q(2)\ q(3) - q(5)\\
&q(3,1,1) = \frac{1}{2}\ (q(1)^2\ q(3) - q(2)\ q(3) - 2q(1)\ q(4) + 2q(5))\\
&q(2,2,1) = \frac{1}{2}\ (q(1)\ q(2)^2 - 2\ q(2)\ q(3) - q(1)\ q(4) + 2\ q(5))\\
&q(2,1,1,1) = \frac{1}{6}\ (q(1)^3\ q(2) - 3\ q(1)\ q(2)^2 - 3\ q(1)^2\ q(3) + \\
&\qquad\qquad 5\ q(2)\ q(3) + 6\ q(1)\ q(4) - 6\ q(5))\\
&q(1,1,1,1,1) = \frac{1}{120}\ (q(1)^5 - 10\ q(1)^3\ q(2) + 15\ q(1)\ q(2)^2 + 20\ q(1)^2\ q(3) - \\
&\qquad\qquad 20\ q(2)\ q(3) - 30\ q(1)\ q(4) + 24\ q(5))
\end{aligned} \tag{A1.5}$$

Beyond order 5, we will give the rules recursively as in the following:

$$
\begin{aligned}
&q(i_1,i_2) = q(i_1)\ q(i_2)\ -\ q(i_1\ +\ i_2) \qquad i_1\ >\ i_2\\
&q(i_1,i_1) = \frac{1}{2}(q(i_1)^2\ -\ q(i_1\ +\ i_1))\\
&q(i_1,i_2,i_2) = q(i_1)\ q(i_2,i_2)\ -\ q(i_1\ +\ i_2,i_2) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_2) = \frac{1}{2}(q(i_1)\ q(i_1,i_2)\ -\ q(i_1\ +\ i_1,i_2)\ -\ q(i_1\ +\ i_2,i_1)) \qquad i_1\ >\ i_2\\
&q(i_1,i_2,i_3) = q(i_1)\ q(i_2,i_3)\ -\ q(i_1\ +\ i_2,i_3)\ -\ q(i_1\ +\ i_3,i_2) \qquad i_1\ >\ i_2\ > i_3\\
&q(i_1,i_1,i_1) = \frac{1}{3}(q(i_1)\ q(i_1,i_1)\ -\ q(i_1\ +\ i_1,i_1))\\
&q(i_1,i_2,i_2,i_2) = q(i_1)\ q(i_2,i_2,i_2)\ -\ q(i_1\ +\ i_2,i_2,i_2) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_1,i_2) = \frac{1}{3}(q(i_1)\ q(i_1,i_1,i_2)\ -\ q(i_1\ +\ i_2,i_1,i_1)) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_2,i_2) = \frac{1}{2}(q(i_1)\ q(i_1,i_2,i_2)\ -\ q(i_1\ +\ i_2,i_1,i_2)) \qquad i_1\ >\ i_2\\
&q(i_1,i_2,i_3,i_3) = q(i_1)\ q(i_2,i_3,i_3)\ -\ q(i_1\ +\ i_2,i_3,i_3)\ -\ q(i_1\ +\ i_3,i_2,i_3) \qquad i_1\ >\ i_2\ > i_3\\
&q(i_1,i_1,i_1,i_1) = \frac{1}{4}(q(i_1)\ q(i_1,i_1,i_1)\ -\ q(i_1\ +\ i_1,i_1,i_1))\\
&q(i_1,i_2,i_2,i_2,i_2) = q(i_1)\ q(i_2,i_2,i_2,i_2)\ -\ q(i_1\ +\ i_2,i_2,i_2,i_2) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_2,i_2,i_2) = \frac{1}{2}(q(i_1)\ q(i_1,i_2,i_2,i_2)\ -\ q(i_1\ +\ i_2,i_1,i_2,i_2)) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_1,i_1,i_1) = \frac{1}{5}(q(i_1)\ q(i_1,i_1,i_1,i_1)\ -\ q(i_1\ +\ i_1,i_1,i_1,i_1))\\
&q(i_1,i_2,i_2,i_2,i_2,i_2) = q(i_1)\ q(i_2,i_2,i_2,i_2,i_2)\ -\ q(i_1\ +\ i_2,i_2,i_2,i_2,i_2) \qquad i_1\ >\ i_2\\
&q(i_1,i_1,i_1,i_1,i_1,i_1) = \frac{1}{6}(q(i_1)\ q(i_1,i_1,i_1,i_1,i_1)\ -\ q(i_1\ +\ i_1,i_1,i_1,i_1,i_1))\\
&q(i_1,i_1,i_1,i_1,i_1,i_1,i_1) = \frac{1}{7}(q(i_1)\ q(i_1,i_1,i_1,i_1,i_1,i_1)\ -\ q(i_1\ +\ i_1,i_1,i_1,i_1,i_1,i_1))
\end{aligned}
$$

APPENDIX. 2

In section.V we show the way of extracting two eigenvalue polinomials which are shown to be valid in 4th order. We repeat here the analysis in orders s=5,6,7 and we give our solutions respectively for

(1) the 4 eigenvalue polinomials in order 7 (=5+2=4+3=3+2+2)

$$
\begin{aligned}
k_7(1,N) &= \frac{k_7(4,N)}{14\ (2\ N^2+4\ N-15)}\ (N^4+4\ N^3+41\ N^2+74\ N+120)\\
k_7(2,N) &= -\frac{k_7(4,N)}{2\ (N+1)\ (2\ N^2+4\ N-15)}\ (N^4+4\ N^3+17\ N^2+26\ N-96)\\
k_7(3,N) &= -\frac{k_7(4,N)}{2\ (N+1)\ (2\ N^2+4\ N-15)}\ (N^4+4\ N^3+5\ N^2+2\ N+60)
\end{aligned}
$$

$$
\begin{aligned}
P_7(\Lambda^+,N) =\ & k_7(1,N)\ \Theta(7,\Lambda^+,N)\\
&+\ k_7(2,N)\ \Theta(5,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)\\
&+\ k_7(3,N)\ \Theta(4,\Lambda^+,N)\ \Theta(3,\Lambda^+,N)\\
&+\ k_7(4,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)^2
\end{aligned} \tag{A2.1}
$$

$$k_{43}(1,N) = -\frac{8640\ k_{43}(5,N)}{(N+1)}\ (N-1)\ N\ (N+2)\ (N+3)\ (N^4+4\ N^3+5\ N^2+2\ N+60)$$

$$k_{43}(2,N) = \frac{8640\ k_{43}(5,N)}{(N+1)^2}\ (N-1)\ (N+3)\ (6\ N^6+36\ N^5+13\ N^4- 188\ N^3-N^2+470\ N+840)$$

$$k_{43}(3,N) = \frac{720\ k_{43}(5,N)}{(N+1)^2}\ (N-1)\ (N+3)\ (N^8+8\ N^7+16\ N^6-16\ N^5+681\ N^4+ 2980\ N^3-986\ N^2-8060\ N-8400)$$

$$k_{43}(4,N) = -\frac{720\ k_{43}(5,N)}{(N+1)}\ (N-1)\ (N+3)\ (2\ N^6+12\ N^5+121\ N^4+ 404\ N^3-957\ N^2-2690\ N+4200)$$

$$\begin{aligned} P_{43}(\Lambda^+,N) = &\ k_{43}(1,N)\ \Theta(7,\Lambda^+,N) \\ &+\ k_{43}(2,N)\ \Theta(5,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\ &+\ k_{43}(3,N)\ \Theta(4,\Lambda^+,N)\ \Theta(3,\Lambda^+,N) \\ &+\ k_{43}(4,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)^2 \\ &+\ k_{43}(5,N)\ \Theta(3,\Lambda^+,N) \end{aligned} \tag{A2.2}$$

$$k_{52}(1,N) = -\frac{24\ k_{52}(6,N)}{g_{52}(N)}\ (N-3)\ (N-2)\ (N-1)\ N\ (N+2)\ (N+3)\ (N+4)\ (N+5)\ (N^4+4\ N^3+17\ N^2+26\ N-96)$$

$$k_{52}(2,N) = \frac{12\ k_{52}(6,N)}{5\ g_{52}(N)(N+1)}\ (N-3)\ (N-2)\ (N-1)\ (N+3)\ (N+4)\ (N+5)\ (N^8+8\ N^7+32\ N^6+80\ N^5+515\ N^4+1676\ N^3+1648\ N^2+72\ N-10080)$$

$$k_{52}(3,N) = \frac{24\ k_{52}(6,N)}{g_{52}(N)\ (N+1)}\ (N-3)\ (N-2)\ (N-1)\ (N+3)\ (N+4)\ (N+5)\ (6\ N^6+36\ N^5+13\ N^4-188\ N^3-N^2+470\ N+840)$$

$$k_{52}(4,N) = -\frac{12\ k_{52}(6,N)}{g_{52}(N)}\ (N-3)\ (N-2)\ (N-1)\ (N+3)\ (N+4)\ (N+5)\ (N^6+6N^5-6N^4-64N^3+281N^2+706N-840)$$

$$k_{52}(5,N) = -\frac{k_{52}(6,N)}{5\ g_{52}(N)}\ (N-5)\ (N-4)\ (N-3)(N-2)\ (N-1)\ N\ (N+1)^2\ (N+2)\ (N+3)\ (N+4)\ (N+5)\ (6+N)\ (N+7)\ (N^2+2\ N+6)$$

$$\begin{aligned} P_{52}(\Lambda^+,N) = &\ k_{52}(1,N)\ \Theta(7,\Lambda^+,N) \\ &+\ k_{52}(2,N)\ \Theta(5,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\ &+\ k_{52}(3,N)\ \Theta(4,\Lambda^+,N)\ \Theta(3,\Lambda^+,N) \\ &+\ k_{52}(4,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)^2 \\ &+\ k_{52}(5,N)\ \Theta(5,\Lambda^+,N) \\ &+\ k_{52}(6,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \end{aligned} \tag{A2.3}$$

$$k_{322}(1,N) = \frac{34560\ k_{322}(7,N)}{g_{322}(N)}\ (N-1)\ N\ (N+1)\ (N+2)\ (N+3)\ (2N^2+4N-15)$$
$$k_{322}(2,N) = -\frac{8640\ k_{322}(7,N)}{g_{322}(N)}\ (N-1)\ (N+3)\ (N^6+6\ N^5-6\ N^4- 64\ N^3+281\ N^2+706\ N-840)$$
$$k_{322}(3,N) = -\frac{1440\ k_{322}(7,N)}{g_{322}(N)}\ (N-1)\ (N+3)\ (2\ N^6+12\ N^5+121\ N^4+ 404\ N^3-957\ N^2-2690\ N+4200)$$
$$k_{322}(4,N) = \frac{720\ k_{322}(7,N)}{g_{322}(N)(N+1)}\ (N-1)\ (N+3)\ (N^8+8\ N^7-3\ N^6-130\ N^5+ 109\ N^4+1452\ N^3+5113\ N^2+6890\ N-4200)$$
$$k_{322}(5,N) = \frac{720\ k_{322}(7,N)}{g_{322}(N)}(N-5)\ (N-4)\ (N-1)\ N\ \ (N+1)\ (N+2) (N+3)\ (N+6)\ (N+7)\ (N^2+2\ N-1)$$
$$k_{322}(6,N) = -\frac{120\ k_{322}(7,N)}{g_{322}(N)}(N-5)\ (N-4)\ (N-1)\ N\ (N+2)\ (N+3)\ (N+6) (N+7)\ (N^4+4\ N^3+6\ N^2+4\ N+25)$$

$$\begin{aligned} P_{322}(\Lambda^+,N) = &\ k_{322}(1,N)\ \Theta(7,\Lambda^+,N) \\ +&\ k_{322}(2,N)\ \Theta(5,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\ +&\ k_{322}(3,N)\ \Theta(4,\Lambda^+,N)\ \Theta(3,\Lambda^+,N) \\ +&\ k_{322}(4,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)^2 \\ +&\ k_{322}(5,N)\ \Theta(5,\Lambda^+,N) \\ +&\ k_{322}(6,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\ +&\ k_{322}(7,N)\ \Theta(3,\Lambda^+,N) \end{aligned} \tag{A2.4}$$

where

$$\begin{aligned} g_7(N) &\equiv \prod_{i=-5}^{7}(N+i) \\ g52(N) &= (N^2+2\ N-1)\ g_7(N) \\ g43(N) &= (N+1)\ g_7(N) \\ g322(N) &= (5\ N^2+10\ N+11)\ g_7(N) \end{aligned}$$

(2) The 4 eigenvalue polinomials in order 6 (=4+2=3+3=2+2+2)

$$k_6(1,N) = -\frac{30240\ k_6(5,N)}{g_6(N)}\ (N^4+4\ N^3+21\ N^2+34\ N+24)$$
$$k_6(2,N) = \frac{181440\ k_6(5,N)}{g_6(N)\ (N+1)}\ (N-1)\ (N+3)\ (N^2+2\ N+6)$$
$$k_6(3,N) = \frac{30240\ k_6(5,N)}{g_6(N)\ (N+1)}\ (3\ N^4+12\ N^3+7\ N^2-10\ N+72)$$
$$k_6(4,N) = -\frac{211680\ k_6(5,N)}{g_6(N)}\ (N^2+2\ N-6)$$

$$
\begin{aligned}
P_6(\Lambda^+,N) = {} & k_6(1,N)\ \Theta(6,\Lambda^+,N) \\
+ {} & k_6(2,N)\ \Theta(4,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\
+ {} & k_6(3,N)\ \Theta(3,\Lambda^+,N)^2 \\
+ {} & k_6(4,N)\ \Theta(2,\Lambda^+,N)^3 \\
+ {} & k_6(5,N)
\end{aligned} \qquad (A2.5)
$$

$$
\begin{aligned}
k_{33}(1,N) &= -\frac{3024\ k_{33}(5,N)}{g_6(N)}\ (3\ N^4+12\ N^3+7\ N^2-10\ N+72) \\
k_{33}(2,N) &= \frac{45360\ k_{33}(5,N)}{N\ (N+1)\ (N+2)\ g_6(N)}\ (N^6+6N^5+5N^4-20N^3-20N^2+16N+96) \\
k_{33}(3,N) &= \frac{1008\ k_{33}(5,N)}{N\ (N+1)\ (N+2)\ g_6(N)}\ (N^8+8N^7-112N^5+127N^4+ \\
&\qquad 1404N^3+580N^2-2032N-3840) \\
k_{33}(4,N) &= -\frac{12096\ k_{33}(5,N)}{N\ (N+2)\ g_6(N)}\ (4\ N^4+16\ N^3-35\ N^2-102\ N+180)
\end{aligned}
$$

$$
\begin{aligned}
P_{33}(\Lambda^+,N) = {} & k_{33}(1,N)\ \Theta(6,\Lambda^+,N) \\
+ {} & k_{33}(2,N)\ \Theta(4,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\
+ {} & k_{33}(3,N)\ \Theta(3,\Lambda^+,N)^2 \\
+ {} & k_{33}(4,N)\ \Theta(2,\Lambda^+,N)^3 \\
+ {} & k_{33}(5,N)
\end{aligned} \qquad (A2.6)
$$

$$
\begin{aligned}
k_{42}(1,N) &= \frac{483840\ k_{42}(8,N)}{g_{42}(N)}\ (N-1)\ (N+3)\ (N^2+2N+6) \\
k_{42}(2,N) &= -\frac{60480\ k_{42}(8,N)}{g_{42}(N)\ N\ (N+1)\ (N+2)}\ (N-1)\ (N+3) \\
&\qquad (N^2+2\ N+6)\ (120+2\ N+5\ N^2+4\ N^3+N^4) \\
k_{42}(3,N) &= -\frac{1209600\ k_{42}(8,N)}{g_{42}(N)\ N\ (N+1)\ (N+2)}(N^6+6\ N^5+5\ N^4-20\ N^3-20\ N^2+16\ N+96) \\
k_{42}(4,N) &= \frac{60480\ k_{42}(8,N)}{g_{42}(N)\ N\ (N+2)}(N-1)(N+3)(2\ N^4+8\ N^3-25\ N^2-66\ N+360) \\
k_{42}(5,N) &= \frac{5040\ k_{42}(8,N)}{g_{42}(N)}\ (N-4)\ (N-3)\ (N+1)^2\ (N+5)\ (N+6)\ (N^2+2\ N+2) \\
k_{42}(6,N) &= -\frac{5040\ k_{42}(8,N)}{g_{42}(N)}\ (N-4)\ (N-3)\ (N+1)\ (N+5)\ (N+6)\ (2\ N^2+4\ N-1) \\
k_{42}(7,N) &= -\frac{84\ k_{42}(8,N)}{g_{42}(N)}\ (N-4)\ (N-3)\ (N-2)\ (N-1)\ (N+1)^2 \\
&\qquad (N+3)\ (N+4)\ (N+5)\ (N+6)
\end{aligned}
$$

$$
\begin{aligned}
P_{42}(\Lambda^+,N) = {} & k_{42}(1,N)\ \Theta(6,\Lambda^+,N) \\
+ {} & k_{42}(2,N)\ \Theta(4,\Lambda^+,N)\ \Theta(2,\Lambda^+,N) \\
+ {} & k_{42}(3,N)\ \Theta(3,\Lambda^+,N)^2 \\
+ {} & k_{42}(4,N)\ \Theta(2,\Lambda^+,N)^3 \\
+ {} & k_{42}(5,N)\ \Theta(4,\Lambda^+,N) \\
+ {} & k_{42}(6,N)\ \Theta(2,\Lambda^+,N)^2 \\
+ {} & k_{42}(7,N)\ \Theta(2,\Lambda^+,N) \\
+ {} & k_{42}(8,N)
\end{aligned} \qquad (A2.7)
$$

$$
\begin{aligned}
k_{222}(1,N) &= -\frac{483840\ k_{222}(8,N)}{g_{222}(N)}\ (N^2+2\ N-6)\\
k_{222}(2,N) &= \frac{51840\ k_{222}(8,N)}{N\ (N+1)\ (N+2)\ g_{222}(N)}\ (N-1)\ (N+3)\\
&\qquad (2\ N^4+8\ N^3-25\ N^2-66\ N+360)\\
k_{222}(3,N) &= \frac{276480\ k_{222}(8,N)}{N\ (N+1)\ (N+2)\ g_{222}(N)}(4\ N^4+16\ N^3-35N^2-102\ N+180)\\
k_{222}(4,N) &= -\frac{8640\ k_{222}(8,N)}{N\ (N+1)^2\ (N+2)\ g_{222}(N)}\ (N^8+8\ N^7-7\ N^6-154\ N^5-79\ N^4+\\
&\qquad 860\ N^3+1777\ N^2+1338\ N-3240)\\
k_{222}(5,N) &= -\frac{4320\ k_{222}(8,N)}{g_{222}(N)}\ (N-4)\ (N-3)\ (N+5)\ (N+6)\ (2\ N^2+4\ N-1)\\
k_{222}(6,N) &= \frac{2160\ k_{222}(8,N)}{(N+1)\ g_{222}(N)}(N-4)\ (N-3)\ (N+5)\ (N+6)\ (N^4+4\ N^3-8\ N+13)\\
k_{222}(7,N) &= -\frac{36\ k_{222}(8,N)}{g_{222}(N)}\ (N-4)\ (N-3)\ (N-2)\ (N-1)\ (N+3)\ (N+4)\\
&\qquad (N+5)\ (N+6)\ (5N^2+10\ N+11)
\end{aligned}
$$

$$
\begin{aligned}
P_{222}(\Lambda^+,N) =\ & k_{222}(1,N)\ \Theta(6,\Lambda^+,N)\\
+\ & k_{222}(2,N)\ \Theta(4,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)\\
+\ & k_{222}(3,N)\ \Theta(3,\Lambda^+,N)^2\\
+\ & k_{222}(4,N)\ \Theta(2,\Lambda^+,N)^3\\
+\ & k_{222}(5,N)\ \Theta(4,\Lambda^+,N)\\
+\ & k_{222}(6,N)\ \Theta(2,\Lambda^+,N)^2\\
+\ & k_{222}(7,N)\ \Theta(2,\Lambda^+,N)\\
+\ & k_{222}(8,N)
\end{aligned}
\tag{A2.8}
$$

where

$$
\begin{aligned}
g_6(N) &\equiv \prod_{i=-4}^{6}(N+i)\\
g_{42}(N) &= (7\ N^2+14\ N+47)\ g_6(N)\\
g_{222}(N) &= (5\ N^2+10\ N+23)\ g_6(N)
\end{aligned}
$$

(3) The 2 eigenvalue polinomials in order 5 (=3+2)

$$
k_5(1,N) = -\frac{k_5(2,N)}{5\ (N^2+2N-1)}\ (N+1)\ (N^2+2N+6)
$$

$$
\begin{aligned}
P_5(\Lambda^+,N) =\ & k_5(1,N)\ \Theta(5,\Lambda^+,N)\\
+\ & k_5(2,N)\ \Theta(3,\Lambda^+,N)\ \Theta(2,\Lambda^+,N)
\end{aligned}
\tag{A2.9}
$$

$$
\begin{aligned}
k_{32}(1,N) &= \frac{72\ k_{32}(3,N)}{g_5(N)\ (N+1)}\ (N-1)\ N\ (N+2)\ (N+3)\ (N^2+2\ N-1)\\
k_{32}(2,N) &= -\frac{12\ k_{32}(3,N)}{g_5(N)\ (N+1)^2}\ (N-1)\ N\ (N+2)\ (N+3)\ (N^4+4\ N^3+6\ N^2+4\ N+25)
\end{aligned}
$$

$$\begin{aligned}P_{32}(\Lambda^+, N) =\ & k_{32}(1, N)\ \Theta(5, \Lambda^+, N)\\ +\ & k_{32}(2, N)\ \Theta(3, \Lambda^+, N)\ \Theta(2, \Lambda^+, N)\\ +\ & k_{32}(3, N)\ \Theta(3, \Lambda^+, N)\end{aligned} \tag{A2.10}$$

where

$$g_5(N) \equiv \prod_{i=-3}^{5} (N + i) \quad .$$